Motor and Failure Mode

The propellant in the example motor is a composite solid consisting of CTPB binder, ammonium perchlorate and aluminum powder. The outside diameter of the spherical grain is 480 mm, and the geometry of the inner free volume of the grain is of the 8-slotted star shape. In order to reduce thermal strain in the grain, the surface between the grain and the chamber is unbounded. The difference between the two motors is in the depth of stress-relief boots. About a day after the cooling procedure was completed, fracture of the propellant occurred due to restrained thermal shrinkage along the inner surface of the grain. Many vacuoles, which appeared as white bands, were observed on the fillets of the inner bore of the grain. This means that the value of stress or strain was on the verge of the critical one at which the failure of the propellant took place. In order to estimate the strains due to thermal shrinkage at the inner fillet, three-dimensional photoelastic analysis[2,3] was carried out with respect to the variations of the depth of the stress-relief boots. The maximum strain concentration factors ($K_\varepsilon = \varepsilon/\varepsilon_0$) on the fillets of the motors A and B were estimated as 4.9 and 1.5, respectively (ε_0 is the tangential strain on the inner free surface of a long cylinder whose web-ratio is the same as the largest one in the spherical motor). It was known through the experimental analysis that the stress state along the fillets could almost be regarded as uniaxial.

Results

The restrained thermal strain of the spherical grains increases with decreasing grain temperature since the strain rate in the propellant should be proportional to the decreasing rate. The actual spherical rocket motors are cooled down at the rate of 0.5°C/hr. If the time-temperature shift factors had been already known, the curves of the ultimate strain could be easily given at any temperature by shifting the master curve along the temperature axis. In Fig. 4 the solid curve corresponds to the statistical line of the minimum strength, and the dotted solid line to the line of the observed minimum strength.

The tangential strains on the inner free surface of the grain whose outer surface is restrained to shrink can be expressed as

$$\varepsilon_p = (1+\mu)\left[(1-\mu)\frac{2K_\varepsilon M^2}{M^2(1-2\mu)+1} - 1\right](\alpha_p - \alpha_c)d\tau \quad (2)$$

where μ is Poisson's ratio of the propellant, dT is the temperature drop of the motor, M is the ratio of the outer to the inner diameters of the grain, α_p and α_c are thermal expansion coefficients of the propellant and the chamber, respectively. The straight lines in Fig. 4 show the strains in the two motors, which were calculated from Eq. (2). At about 30°C, the maximum strain of motor A (Fig. 1) reaches the master curve of the mean of ε_m. This corresponds to the fact that the white band (many vacuoles) appeared along the fillets of motor A when the motor was taken out from the cooling pit. On the other hand, any failure of the propellant of motor B did not occur when the maximum strain reached the line of the observed minimum strength at room temperature.

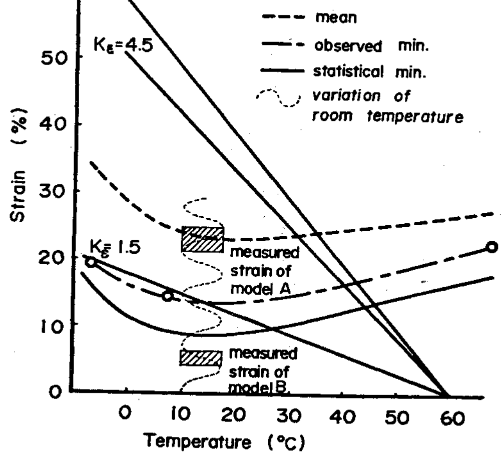

Fig. 4 Ultimate strain and calculated strain vs temperature.

If it is possible in a practical sense to adopt the statistical minimum value as the critical one for design of motors, this value will be most suitable because the safety of the propellant grain will be ensured statistically. It should be noticed in Fig. 3 that the statistical minimum values of the ultimate strength are close to the observed minimum values and are not so small in comparison with the mean of the ultimate strength. Therefore, the following relation is applicable as a practical design criterion

$$\varepsilon_{cr} = \varepsilon \quad (3)$$

where ε_{cr} is the design critical strain.

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Conditions for Stability of an Ablating Symmetric Rolling Re-Entry Vehicle

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Nomenclature

A, B, C	= coefficients defined in Eq. (1)
$\bar{C}$	= a quantity defined in Eq. (8)
C_D	= drag coefficient
$C_{N\alpha}$	= normal force coefficient derivative
$C_{M\alpha}$	= pitching moment derivative, $\partial C_M/\partial\alpha$
$C_{Ma\alpha}$	= ablation induced pitching moment derivative
$C_{m\dot{\alpha}}$	= moment due to rate of angle-of-attack derivative
C_{mq}	= damping in pitch derivative
$C_{np\alpha}$	= Magnus moment derivative
D	= a coefficient defined in Eq. (1)
$G(s), G_1(s), G_2(s)$	= functions defined in Eq. (20)
g	= a "perturbation" vector
H	= altitude
I, I_x	= transverse and axial moments of inertia
k_a, k_T	= radii of gyration, defined in Eq. (1)
l	= a characteristic length
m	= mass
P	= normalized rate of roll, pl/V

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p	= rate of roll
P, Q	= matrices of quadratic forms v and v' respectively
R	= a matrix, defined by Eq. (21)
S	= reference area
s	= a nondimensional independent variable, $(1/l)\int_0^t V dt$
t	= time
u	= a variable
V	= velocity
v	= Liapunov's function
x	= a variable
α	= angle of attack
β	= atmospheric density exponent
γ	= angle of pitch
δ	= a complex angle of attack, $\delta_1 + i\delta_2$
λ	= a characteristic value
ρ	= atmospheric density
Σ	= an expression defined in Eq. (7)
σ	= a path lag due to ablation
ω	= derivative of $\delta = d\delta/ds$; $(\omega = \omega_1 + i\omega_2)$
$(\ )'$	$= d(\ )/ds$

Subscripts

m	= mean
o	= initial
max	= maximum
1, 2	= indices of the real and imaginary parts of δ and ω

Superscripts

T	= transposed

1. Introduction

In a recent study, Burton and Abel[1] considered the effect of an additional pitching moment due to injection of the ablative material into the vehicle's boundary layer. They argued that such an additional moment depends on the value of the complex angle of attack, which is out of phase with the vehicle's motion, or in other words, lags behind the present state. The analysis[1] was confined to the case of a constant dynamic pressure and the effect of the time lag on natural frequencies and damping parameter was investigated.

In the present paper the emphasis is on obtaining a conservative estimate of the time lag for which the vehicle's motion is stable. Liapunov's function is used to obtain a differential inequality to which the comparison principle is applied. By this method the desired stability criterion is obtained.

2. Analysis

Consider the differential equation governing the motion of an ablating spinning re-entry vehicle, in body axes

$$\delta'' + (A + iB)\delta' + (C + iD)\delta = K\delta(s - \sigma), \ \sigma > 0 \qquad (1)$$

where the coefficients A,B,C,D and K are

$$A = (\rho Sl/2m)[C_{N_\alpha} - 2C_D - k_T^{-2}(C_{m_q} + C_{m\dot{\alpha}})]$$

$$B = -P[(I_x/I) + 2]$$

$$C = -(\rho Sl/2m)k_T^{-2}C_{M_\alpha} - P^2[(1 - (I_x/I)]$$

$$D = -P I_x \frac{\rho Sl}{2Im}[C_{N_\alpha} - C_D - k_a^{-2}C_{np\alpha}] + P\frac{\rho Sl}{2m}[C_{N_\alpha} - 2C_D - k_T^{-2}(C_{mq} + C_{m\dot{\alpha}})]$$

$$K = -(\rho Sl/2m)k_T^{-2}C_{M\alpha\alpha}$$

and

$$k_T = (I/ml^2)^{1/2}, \ k_a = (I_x/ml^2)^{1/2}$$

The right-hand side of Eq. (1) consists of the additional pitching moment due to ablation, which depends on the angle of attack occuring σ calibers (units of s) prior to the current state. This term can be rewritten as follows.

Using the mean value theorem we get

$$\delta(s) - \delta(s - \sigma) = \sigma\delta'(s - \sigma_m)$$

where

$$0 < \sigma_m < \sigma \qquad (2)$$

Introducing the value of $\delta(s - \sigma)$ from Eq. (2) into Eq. (1), the following equation is obtained

$$\delta'' + (A + iB)\delta' + (C - K + iD)\delta = K\sigma\delta'(s - \sigma_m) \qquad (3)$$

Equation (3) will be referred to as the "augmented" equation of motion.

The variation of atmospheric density will be expressed through the "exponential approximation"

$$\rho = \rho_0 e^{-\beta H} \qquad (4)$$

If the changes in γ and in the velocity, V are much slower than the changes in δ, Eq. (4) can be expressed in terms of the variable s

$$\rho = \rho_0 e^{-(\beta l \sin\gamma)s} \qquad (5)$$

Now, A, C and D are all functions of ρ, as can be seen from Eq. (1). The term B will be assumed to remain constant, as P is independent of ρ in many practical cases.

Equation (3) will now be rewritten in the state variable form

Let

$$\delta = \delta_1 + i\delta_2$$

then

$$\left.\begin{aligned} \delta_1' &= \omega_1 \\ \delta_2' &= \omega_2 \\ \omega_1' &= -(C - K)\delta_1 + D\delta_2 - A\omega_1 + B\omega_2 \\ &\quad + K\sigma\omega_1(s - \sigma_m) \\ \omega_2' &= -D\delta_1 - (C - K)\delta_2 - B\omega_1 - A\omega_2 + \\ &\quad K\sigma\omega_2(s - \sigma_m) \end{aligned}\right\} \qquad (3')$$

The stability of re-entry motion will now be investigated.

In Ref. (2) it was shown that the following function is a Liapunov function for the "augmented" equation with $\sigma = 0$

$$v = [(A^2 + B^2)/2 + C - K](\delta_1^2 + \delta_2^2) + (\omega_1^2 + \omega_2^2) + \\ + A(\delta_1\omega_1 + \delta_2\omega_2) + B(\delta_1\omega_2 - \delta_2\omega_1) \qquad (6)$$

It can be shown that if

$$\Sigma \triangleq (A^2 + B^2)/4 + C - K > 0 \qquad (7)$$

then

$$v \geq v_{(s=0)} > 0$$

and thus v is positive definite.

The derivative of v along the solutions of Eq. (3′) is

$$v' = -[A(C - K) + BD + (A^2 + \tilde{C} - K)\beta l \sin\gamma] \times \\ \times (\delta_1^2 + \delta_2^2) - A(\omega_1^2 + \omega_2^2) - 2D(\delta_1\omega_2 - \delta_2\omega_1) - \\ - A\beta l \sin\gamma(\delta_1\omega_1 + \delta_2\omega_2) \qquad (8)$$

where

$$\tilde{C} = -(\rho Sl/2m)k_T^{-2}C_{M_\alpha} = C + P^2[1 - (I_x/I)]$$

It is negative semidefinite if

$$\left.\begin{aligned} &A > 0 \\ \text{and}\quad &A^2(C - K) + ABD - D^2 + A(A^2 + \tilde{C} - K) \times \\ &\quad \times \beta l \sin\gamma - (A^2/4)(\beta l \sin\gamma)^2 > 0 \end{aligned}\right\} \qquad (9)$$

According to Hahn[3] the motion is stable (in the sense of Liapunov) if Eqs. (7) and (9) hold.

Both v and v' are quadratic forms. Therefore, they can be related to each other using the theory of pencils of quadratic forms (see e.g. Ref. 4).

Let

$$v = x^T P(s) x$$
$$v' = x^T Q(s) x$$

where

$$x \equiv [\delta_1, \delta_2, \omega_1, \omega_2]^T$$

$P(s)$-a symmetric positive definite matrix and $Q(s)$-a symmetric matrix.

Let λ be a solution of $\mathrm{Det}(Q - \lambda P) = 0$.

Then

$$v' \leq \lambda_{max}(s) v$$

For v and v' given by Eq. (6) and (8), respectively,

$$\lambda_{max} = -\tilde{A} + \{\tilde{A}^2 - (1/\Sigma)[\tilde{A}^2(C-K) + ABD - D^2 + A(A^2 + \tilde{C} - K)\beta l \sin\gamma - (A^2/4)(\beta l \sin\gamma)^2]\}^{1/2}$$

where

$$\tilde{A} = A + \frac{[(A^2/2) + (\tilde{C} - K)]\beta l \sin\gamma}{2\Sigma} \tag{10}$$

Next the v-function, Eq. (6), will be applied to investigate the stability of motion given by Eq. (3′) for $\sigma \neq 0$.

The derivative v' is now

$$v'_{\sigma \neq 0} = v'_{\sigma = 0} + \nabla v^T \cdot g \tag{11}$$

$$R = \begin{bmatrix} -e^{-G_2(s)} G_1'(s) - \mu & e^{-G_2(s)} \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma \\ e^{-G_2(s)} \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma & \mu \end{bmatrix} \tag{21}$$

where g is the perturbation vector given by the following equation

$$g = K\sigma[0, 0, \omega_1(s - \sigma_m), \omega_2(s - \sigma_m)]^T \tag{12}$$

An upper bound on $v'_{\sigma \neq 0}$ can be expressed as

$$v'_{\sigma \neq 0} \leq \lambda_{max}(s) v + \|\nabla v^T\| \cdot \|g\| \tag{13}$$

It is easy to show that for v given by Eq. (6)

$$\|\nabla v^T\| \cdot \|g\| \leq 2(v)^{1/2} \cdot \|g\| \tag{14}$$

Thus for Eq. (3), the derivative of v is bounded as follows

$$v' \leq \lambda_{max}(s) v + 2(v)^{1/2} \sigma |K| [\omega_1^2(s - \sigma_m) + \omega_2^2(s - \sigma_m)]^{1/2} \tag{15}$$

In order to obtain an expression in v only, we have to find the relationship between $\|\omega\|$ and v. Here again the quadratic form $\|\omega\|^2$ is related to v according to the theory of pencils of quadratic forms.[5]

The resulting relation is

$$\|\omega\|^2 \leq \frac{(A^2 + B^2)/2 + C - K}{\Sigma} v \tag{16}$$

Introducing this result into Eq. (15) we get a differential inequality in v

$$v' \leq \lambda_{max}(s) v + 2(v)^{1/2} \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma [v(s - \sigma_m)]^{1/2} \tag{17}$$

Dividing both sides of this inequality by $(v)^{1/2}$ and defining

$$u = (v)^{1/2}$$

we arrive at

$$u' \leq \tfrac{1}{2} \lambda_{max}(s) u + \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma u(s - \sigma_m) \tag{18}$$

The comparison principle[6] will now be applied to the inequality (18). Let $r(s)$ be the solution of

$$r' = \tfrac{1}{2}\lambda_{max}(s) r + \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma r(s - \sigma_m) \tag{19}$$

with

$$r(s_0) = u(s_0)$$

Then

$$r(s) \leq u(s) \quad \text{for} \quad s > s_0$$

Therefore, if the equilibrium of Eq. (19) is stable u approaches zero and as $u = (v)^{1/2}$ the solution of Eq. (3′) is stable.

The stability of the equilibrium of Eq. (19) can be established adapting the results given by Sinha.[7]

Let

$$G(s) = \int_0^s \lambda_{max}(t)\, dt = G_1(s) + G_2(s) \tag{20}$$

where $G_1'(s)$ is bounded and negative and $G_2(s)$ is bounded.

According to Sinha's criteria, the equilibrium of Eq. (19) is stable if there exists a positive μ such that the following matrix is positive definite

The matrix R is positive definite if

$$-e^{-G_2(s)} G_1'(s) > \mu > 0$$

and

$$(-e^{-G_2(s)} G_1'(s) - \mu)\mu - e^{-2G_2(s)} \left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right) K^2 \sigma^2 > 0 \tag{22}$$

The first of the inequalities (22) is satisfied if $G_1'(s)$ is negative, as required. The second of these inequalities implies that a positive μ will exist if

$$|G_1'(s)| > 2\left(\frac{(A^2 + B^2)/2 + C - K}{\Sigma}\right)^{1/2} |K| \sigma, \ Vs \tag{23}$$

Thus a conservative estimate of the upper bound on σ is given by

$$\sigma \leq \frac{|G_1'(s)| \Sigma^{1/2}}{2[(A^2 + B^2)/2 + C - K]^{1/2} |K|} \tag{24}$$

3. Conclusion

A conservative estimate of the maximum time lag which will render a stable re-entry motion of a symmetric rolling vehicle was found, using Liapunov's function and related techniques. The obtained information is valuable in the initial stages of design of re-entry vehicles, when the aerodynamic quantities, geometry and initial conditions of re-entry are not established with adequate accuracy. The method described here makes use of the knowledge of stability properties of the nonablating missile, to establish stability of the ablating missile. This approach gives simpler stability criteria than those obtained by directly applying Sinha's method to the equations of motion.

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Optimum Rectangular Radiative Fins Having Temperature-Variant Properties

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Nomenclature

a = parameter, Eq. (5)
b = fin thickness
B = beta function
$B_{( )}$ = incomplete beta function
k = thermal conductivity
k_0 = parameter, Eq. (4)
L = fin length
n = parameter, Eq. (4)
q = heat-transfer rate
T = temperature
x = coordinate
ε = emissivity
ε_0 = parameter, Eq. (5)
θ = transformed temperature
ξ = dimensionless coordinate
η = effectiveness
σ = Stefan-Boltzmann constant
ψ = dimensionless temperature
ψ_r = temperature ratio, θ_L/θ_b

Subscripts

b = fin base
L = fin tip
opt = optimum value
s = equivalent environment

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Introduction

THEORETICAL and experimental approaches to the problem of radiative heat transfer from fins possessing constant properties has been the subject of considerable study in recent years. However, a slightly more complicated state of affairs occurs when the properties are affected by temperature. As a result, only few investigations have appeared in the heat-transfer literature. Nevertheless, in spite of these unavoidable obstacles, Stockman and Kramer[1] provided numerical solutions for the temperature field of rectangular radiating fins whose properties vary linearly with temperature. Hung and Appl[2] developed a procedure which yields bounding functions for the temperature distribution of convective-radiative fins of arbitrary profiles. They considered properties that are linear functions of temperature. Shouman[3] derived an expression for the temperature variation of the previous problem extended to the case where the temperature-dependent properties are nonlinear.

Briefly, the present investigation is concerned with the conduction-radiation process in rectilinear fins of constant cross section dissipating heat to the surroundings at constant equivalent temperature. The fin material possesses properties that vary with temperature in a power fashion. This model corresponds to the fitting of piecewise continuous functions of the property-temperature curves.[4] The prime objective of this study is to illustrate a mathematical scheme for obtaining a solution for the radiative heat transfer along the aforesaid fins. This can be achieved by means of an algebraic transformation. Essentially, its aim is to convert the nonlinear differential equation to another of simpler form. Even though the resulting equation is still nonlinear, it may be manageable by available methods in a much easier way. In order to exemplify this, the case where the equivalent environment temperature becomes zero is analyzed. It presents an interesting and peculiar feature inasmuch as the form of the transformed differential equation is similar to the equation that results when the properties are constant. Therefore, the transformation is coupled with the results presented by Liu,[5] who solved the fin problem where the properties are invariant with temperature.

Formulation of the Problem

Consideration is focused on a rectangular thin fin of finite length L and thickness b. Heat is rejected by radiation to an environment whose equivalent temperature is T_s. It is assumed that the radiant interchange between the fin and its base is negligible. Accordingly, the temperature field $T(x)$ at any point along the fin satisfies the energy equation

$$(d/dx)(k\, dT/dx) - (2\varepsilon\sigma/b)(T^4 - T_s^4) = 0 \tag{1}$$

in addition to the boundary conditions

$$T(0) = T_b \tag{2}$$

$$dT(L)/dx = 0 \tag{3}$$

The temperature variation of the thermal properties is represented by means of the relations

$$k = k_0 T^n \tag{4}$$

$$\varepsilon = \varepsilon_0 T^a \tag{5}$$

in the temperature interval under consideration $[T_L, T_b]$.